The Banach-Tarski Paradox

Francis Edward Su

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Harvard University

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Author's note: This paper was originally written for my *Minor Thesis* requirement of the Ph.D. program at Harvard University. The object of this requirement is to learn about a body of work that is generally not in one's own field, in a short period of time (3 weeks), and write an article about it. I chose this topic because I thought it would be quite interesting to learn, and even more fun to tell others about.

While I never published this expository article, I did make it available on my website: `http://www.math.hmc.edu/~su/papers.html`.

If you like it, do write me and let me know. I can be reached at: Department of Mathematics, Harvey Mudd College, Claremont, CA 91711, or by e-mail: `su@math.hmc.edu`.

Introduction

In 1924, S. Banach and A. Tarski proved a truly remarkable theorem: given a solid ball in $\mathbf{R}^3$, it is possible to partition it into finitely many pieces and reassemble them to form *two* solid balls, each identical in size to the first.

At first such a duplication seems patently impossible. However, a moment's reflection reminds us that the mathematical world doesn't always obey intuition.

The intention of this paper is to provide a self-contained exposition of the proof of the Banach-Tarski Paradox and to introduce related topics. We will show, in fact, that the minimal number of pieces in the paradoxical decomposition is five, and prove the stronger form of the Banach-Tarski Paradox : that any two bounded subsets of the plane with non-empty interior are such that one can be "cut" into a finite number of pieces and reassembled to form the other!

Most of the material we will cover is presented in [W], and we shall follow it closely. However, we shall be less ambitious in overall scope and more mindful of continuity of theme. In some places we will provide slightly modified proofs. If anything, the author's hope is that this paper also gives the reader an inkling of how to motivate the results presented herein.

The Axiom of Choice, it will be noted, is indispensable to the proof of the Banach-Tarski paradox. However, as we shall very soon see, there are striking paradoxes which do not use Choice. Hence, the philosophy adopted in ths paper will be the unquestioned acceptance of Choice as a useful foundation in our work, so we will not bow to the pressure to acknowledge which results do or do not depend on the Axiom of Choice.

The reader, it will be assumed, has had a course in abstract algebra and the rudiments of measure theory and analysis. Nevertheless, we establish some terminology: G_n will denote the isometry group of $\mathbf{R}^n$, and SO_n will denote the group of rotations of R^n. A **semigroup** is a set with an associative binary operation and an identity. A **choice set** for a collection of sets (which exists by the Axiom of Choice) will be a set obtained by picking one element from each set in the collection.

Paradoxical Decompositions

Before we set out to prove the Banach-Tarski Paradox , we shall investigate somewhat simpler paradoxes to understand and motivate the kinds of approaches that might be brought to bear on proving theorems of this sort.

One of the earliest paradoxes arose out of grappling with the notion of infinity. The set of integers can be put in 1-1 correspondence with all integers, for instance. This seems strange, at first: if set bijections are interpreted as "equality" in some sense, then $\mathbf{Z}$ is "equal" to a subset of itself. In fact, $\mathbf{Z}$ may be partitioned into two sets (even and odd integers), each "equal" to all of $\mathbf{Z}$. G. Cantor worked with this idea, and developed a theory of cardinality which mathematicians today have few problems understanding or accepting as "intuitive".

Because of this theory, we know that the "number" of points in an interval is the same as the "number" of points in a square. We cannot be too surprised by this, after all, because we have allowed our points to move where they wish as independent "pieces" in our decomposition. Or consider "stretching" $[0, 1)$ — so the number of points in $[0, 1)$ is equal to that in $[0, 2)$, which we can see using the bijection $f(x) = 2x$. Again the familiar paradox emerges:

$[0, 2) = [0, 1) \cup [1, 2)$, the union of two sets each of which is "equal" to the original by a "stretching" bijection. But we are not very shocked that we can stretch $[0, 1)$ to get $[0, 2)$.

However, when we restrict the number of pieces to be finite and the allowable transformations to be isometries of the ambient space, any paradox that persists is hightly counterintuitive.

Notice that the previous paradoxes depended on the set of allowable transformations. Hence we shall demand that our definition of paradoxical be dependent on a group whose action on the set produces the transformations.

Definition 1 *Let G be a group acting on a set X and suppose $E \subseteq X$. E is said to be G-**paradoxical** if for some m,n there exist $g_1, ..., g_m$ and $h_1, ..., h_n \in G$ and pairwise disjoint $A_1, ..., A_m$ and $B_1, ..., B_n \subseteq E$ such that $E = \cup g_i(A_i) = \cup h_j(B_j)$.*

When the isometry group is omitted, we understand it to be the isometry group of X. Note that in our definition, $\{A_i\} \cup \{B_i\}$ may not cover all of E, and the $\{g_i(A_i)\}$ (or $\{h_i(B_j)\}$) may not be pairwise disjoint. This distinction is artificial, however, because we may take the A_i, B_j to be smaller to ensure the pairwise disjointness, and we will prove later that we can use all of E in the decomposition.

Theorem 1 (The Banach-Tarski Paradox) *Any ball in $\mathbf{R}^3$ is paradoxical.*

Paradoxes first emerged in the study of measures. In fact, they were constructed to show the non-existence of certain kinds of measures, such as in the following example.

Theorem 2 *S^1 is countably SO_2-paradoxical (i.e., paradoxical with a countable number of pieces).*

Proof. Consider the RSO_2, the subgroup of SO_2 generated rotations of rational multiples of 2π radians. Let H be a choice set for the cosets of SO_2/RSO_2. Now let $M = \{\sigma(1,0) : \sigma \in H\}$.

Since RSO_2 is countable, we may enumerate it by ρ_i. Let $M_i = \rho_i(M)$. Then $\{M_i\}$ is a countable partition of S^1, and moreover, all the M_i are congruent to each other by rotation.

Hence, each set in $\{M_2, M_4, M_6, ...\}$ may be individually rotated (say, by g_{2i}) to yield $\{M_1, M_2, M_3, ...\}$ whose union is S^1. We can clearly do the same for $\{M_i : i \text{ odd}\}$. Thus S^1 is countably SO_2-paradoxical.

Q.E.D.

Corollary 2.1 *There is no countably additive rotation-invariant measure of total measure* 1 *defined for all subsets of* S^1.

Proof. If there were, then, in the proof above, for $A =$ union of the even-indexed M_i, and $B =$ union of odd-indexed M_i, we would have: $1 = \mu(S^1) = \mu(A) + \mu(B) = \mu(S^1) + \mu(S^1) = 2$.

Corollary 2.2 *There is no countably additive, translation-invariant measure defined on all subsets of* $\mathbf{R}^n$ *and normalizing* $[0,1]^n$. *Hence, there is a subset of* $[0,1]$ *which is not Lebesgue-measurable.*

Proof. It suffices to show such a measure does not exist for $\mathbf{R}^1$, for any such measure μ' in $\mathbf{R}^n$ would induce a measure μ in $\mathbf{R}^1$ by $\mu(A) = \mu'(A \times [0,1]^{n-1})$.

But for μ on $\mathbf{R}^1$, its restriction to subsets of $[0,1)$ would be invariant under G =translations mod 1 and defined on all subsets. Since SO_2's action on S^1 is isomorphic to G's action on $[0,1)$ by the bijection $j : (\cos\theta, \sin\theta) \to \theta/2\pi$, we see that $[0,1)$ is G-paradoxical, which is a contradiction.

One non-Lebesgue-measurable subset [0,1] is $j(M)$, for j as above, and M as in the proof of the theorem.

Q.E.D.

Equidecomposability

The next example we shall discuss is not quite a paradox in the technical sense, but it is rather interesting and will be of use to us later. We shall show that a broken circle ($S^1 \setminus \{pt\}$) can be partitioned into two sets, and reassembled to form a complete circle.

First we need some language to describe this.

Definition 2 *Let G act on a set X, and let $A, B \subseteq X$. A and B are said to be G-**equidecomposable** if A and B can be partitioned into $A_1, ..., A_n$ and $B_1, ..., B_n$ such that each A_i is **congruent** to B_i (there is a $g_i \in G$ such that $g_i(A_i) = B_i$).*

We write $A \sim_G B$, or $A \sim B$.

This means that A can be partitioned into a *finite* number of subsets and reassembled to form B. This is easily seen to be an equivalence relation. Moreover, we see that if $A \sim B$ and A is paradoxical, so is B.

Theorem 3 *$S^1 \setminus \{pt\}$ is equidecomposable to S^1.*

By omission of the group we understand it to be the isometry group of $\mathbf{R}^2$, and we identify S^1 with $\{x : |x| = 1\}$.

Proof. Consider $\mathbf{R}^2$ identified with $\mathbf{C}$, the complex plane. Let the pt be $1 = e^{i0}$. Let $A = \{e^{in} : n \in \{1, 2, 3, ...\}\}$, and let $B = (S^1 \setminus \{pt\}) \setminus A$ be everything else. The points e^{in} are unique, since 2π is irrational. Then, leaving B fixed, rotate A by 1 radian— i.e., use the isometry $\rho(z) = e^{-iz}$. This rotation sends each point "back" one radian, by which we obtain a complete circle.

Q.E.D.

The above construction gives us an interesting idea for the construction of paradoxes. Here we considered the image of a point under the subsemigroup generated by ρ (that is, only positive rotations). Because the semigroup was free, we could use the inverse rotation (obtained from the group containing this semigroup) to "shift" everything back.

In fact, this idea forms the basis of the following:

Theorem 4 (Sierpiński-Mazurkiewicz Paradox) *There exists a subset of* $\mathbf{R}^2$ *which is paradoxical.*

Proof. We can actually explicitly say what that subset is, namely

$$E = \{a_0 + a_1 e^{i\theta} + a_2 e^{2i\theta} + ... + a_n e^{ni\theta} : n, a_j \in \mathbf{N}\}$$

for any θ where $e^{i\theta}$ is transcendental. (In particular, $\theta = 1$ works.)

Our aim is to identify two isometries τ, ρ which generate a subsemigroup S of the isometry group G_2 of the plane (considered as $\mathbf{C}$). Then we shall choose a point x in the plane and consider $E = \{w(x) : w \in S\}$, all the images of this point under words in S.

If for all $a, b \in E, \tau(a) \neq \rho(b)$, then we would see that E is paradoxical, for then $\tau(E) \cap \rho(E) = \emptyset$, and so $\tau^{-1}(\tau(E)) = \rho^{-1}(\rho(E)) = E$.

So choose $\rho(z) = rz$, where $r = e^{i\theta}$ is a transendental complex number. Such a θ exists, since there are only countably many algebraic numbers on the unit circle. Now choose $\tau(z) = z + 1$. These are both isometries of the plane. Let $x = 0$.

Then for any two words $w_1, w_2 \in S$, we shall show that $\tau(w_1(0)) \neq \rho(w_2(0))$, which will show that the words themselves are not equal.

But it is easy to verify that $\tau(w_1(0)) \neq \rho(w_2(0))$ by explicitly writing out both sides of the equation as a polynomial in r, and observing that if they were equal, their difference produces a non-trivial algebraic relation for r, contradicting the choice of r.

Q.E.D.

A few remarks are in order. Since the set E is countable, this theorem says nothing surprising in terms of Lebesgue measure: 2(0)=0. And, since the set has been explicitly defined, we have not used the Axiom of Choice (which supports the view that Choice is not to blame for the phenomenon of paradoxes).

Notice that from $\tau(w_1(0)) \neq \rho(w_2(0))$ we can conclude that S is a *free* subsemigroup, since if distinct reduced words w_1, w_2 are equal, then left cancellation will yield either $\rho(w_1') = \tau(w_2')$ (which must agree on 0, contradicting assumption) or $1 = w'$ (for which one of $\tau(0) = w'(\tau(0))$ or $\rho(0) = w'(\rho(0))$ contradicts the assumption). Hence, freeness seems to play a big role in producing paradoxes, as we shall see in the next section.

An open question related to the Sierpiński-Mazurkiewicz Paradox is whether any *bounded* subset of the plane is paradoxical. It is known that any such subset cannot be paradoxical using two pieces.

We note that the three previous examples of paradoxes involved finding one in a group or a subsemigroup and then in some manner "lifting" the paradox to set on which it acts. Thus it is natural to study paradoxical *groups*, where the group acts on itself by left multiplication.

Paradoxical Groups

The primary example of a paradoxical group is the free group on two generators (sometimes called a free group of rank 2). Recall that a free group F on two generators a, b is the group of all finite words in $a^{\pm 1}, b^{\pm 1}$ with no adjacent pairs of inverse letters, and the operation is juxtaposition with removal of possibly any adjacent pairs of inverse letters.

Theorem 5 *The free group F on two generators σ, τ is F-paradoxical.*

Proof. Let $B(\rho) = \{\text{words beginning on the left with } \rho\}$, where ρ may be $\sigma, \sigma^{-1}, \tau, \tau^{-1}$.

Then $F = \{1\} \cup B(\sigma) \cup B(\sigma^{-1}) \cup B(\tau) \cup B(\tau^{-1})$, where all sets are pairwise disjoint.

But $F = B(\sigma) \cup \sigma B(\sigma^{-1})$ and $F = B(\tau) \cup \tau B(\tau^{-1})$, which shows that F is indeed paradoxical.

We can actually choose to partition F so that all four pieces in the paradoxical decomposition cover F, i.e., $\{1\}$ is included in one of the four sets. To see this, put 1 in $B(\sigma)$. Then we must remove σ^{-1} from $B(\sigma^{-1})$ or else we will duplicate 1 in $\sigma B(\sigma^{-1})$. Inductively we see that σ^{-n} must be removed from $B(\sigma^{-1})$ so as not to duplicate σ^{-n+1}. So let our new partition be the same as the old one except for: $B'(\sigma) = B(\sigma) \cup \{1\} \cup \{\sigma^{-n} : n \in \mathbf{N}^+\}$, and $B'(\sigma^{-1}) = B(\sigma^{-1}) \setminus \{\sigma^{-n} : n \in \mathbf{N}^+\}$. These sets satisfy the same relation $F = B'(\sigma) \cup \sigma B'(\sigma^{-1})$.

Q.E.D.

To ensure that the paradox "lifts" to disjoint well-defined sets, we need a condition on G's action, namely, that no element but the identity fixes any points of the set.

Theorem 6 *Suppose G is a paradoxical group acting on X without nontrivial fixed points. Then X is G-paradoxical.*

Proof. Let $A_i, B_j \subseteq G$ and $g_i, h_j \in G$ be the sets and transformations "witnessing" that G is paradoxical. Take a choice set M from the G-orbits in X. Now $\{g(M) : g \in G\}$ partitions X because G acts without nontrivial fixed points.

Let $A'_i = \cup\{g(M) : g \in A_i\}$ and $B'_j = \cup\{g(M) : g \in B_j\}$. Then $\{A'_i\}$ and $\{B'_j\}$ are all pairwise disjoint, and we can see that $X = \cup\{A'_i\} = \cup\{B'_j\}$.

Q.E.D.

Notice that this idea was used in a subtle way in Theorem 2, where the group acting on S^1 was RSO_2, the orbits were the cosets of SO_2/RSO_2, and the implicitly used paradox on RSO_2 was that this countable group was transitive, and so could be enumerated and decomposed into "even" and "odd" parts as we did for the images of M in the proof of that theorem.

We are now faced with a very interesting question: which groups are paradoxical? The answer to this lies in the investigation of groups bearing a finitely additive measure of total measure 1, defined on all subsets, which is invariant under left multiplication. Such groups are called **amenable** groups, it turns out that they coincide precisely with the non-paradoxical groups. We shall talk a little more about amenable groups later.

More immediately, Theorem 6 tells us a little about which groups are paradoxical.

Corollary 6.1 *If a group G contains H, a paradoxical subgroup, then G is paradoxical.*

Proof. H acts on G by left multiplication, without nontrivial fixed points. (Inverses prevent this from happening.) So G is H-paradoxical by Theorem 6. But then G is G-paradoxical.

Q.E.D.

This corollary immediately implies the next:

Corollary 6.2 *Any group with a free subgroup of rank 2 is paradoxical.*

A stronger converse of Theorem 6 is true:

Theorem 7 *If X is G-paradoxical, then G is paradoxical.*

Proof. . Suppose A'_i, B'_j and g_i, b_j witness that X is G-paradoxical. Then consider H, some G-orbit on X, and a point $x \in H$. Define $A_i = \{g : g(x) \in H \cap A'_i\}$ and $B_j = \{g : g(x) \in H \cap B'_j\}$. These are all pairwise disjoint, since the A_i, B_j are all pairwise disjoint.

$\cup g_i(A_i) = \cup\{g_i g : g(x) \in H \cap A'_i\} = \cup\{g_i g : g_i g(x) \in H \cap g_i(A'_i)\} = \{g' : g'(x) \in X\} = G$

Similarly, we obtain $\cup h_j(B_j) = G$.

Q.E.D.

Our "lifting" technique is not yet overwhelmingly helpful, since group actions often have many nontrivial fixed points. Of course, if we restrict our attention to only the nontrivial fixed points of a space, we can conclude the existence of a paradoxical subset.

We shall be interested in ways to applying Theorem 6 where G=F, the free group of rank 2, and realizing F as a subgroup of G_3, the isometry group of $\mathbf{R}^3$. Then, once we have figured out what to do with the nontrivial fixed points, we shall have proved the Banach-Tarski Paradox .

The Hausdorff Paradox

In 1914, in an attempt to show the nonexistence of certain measures on the sphere, F. Hausdorff constructed the following paradox: Except for a countable number of points, S^2 may be partitioned into three subsets A, B, C such that $A \simeq B \simeq C \simeq (B \cup C)$, where $\simeq$ denotes congruence (not to be confused with $\sim$, which denotes equidecomposability).

His method was to find two rotations of S^2 which generated a free group isomorphic to $\mathbf{Z}_2 * \mathbf{Z}_3$. These were rotations of $2\pi/3$ and π about axes whose interior angle was some θ where $\cos 2\theta$ is transcendental.

Since we are primarily interested in proving the Banach-Tarski Paradox , we shall take a slightly more direct tack, due to Swierczkowski [Sw]. It will suffice to find two rotations which generate which generate a free subgroup of rank 2 of SO_3. This will enable us to obtain a paradoxical decomposition by lifting to S^2. (In Hausdorff's example, $\mathbf{Z}_2 * \mathbf{Z}_3$ also contains such a subgroup.)

Theorem 8 *There exist two independent rotations ϕ, ρ which fix the origin in $\mathbf{R}^3$. Hence, for $n \geq 3$, SO_n contains a free subgroup of rank 2.*

As it turns out, "most" pairs of rotations are independent, and in the following proof arccos(3/5) can be replaced by arccos(r) where r is any rational $\neq 0, \pm(1/2), \pm 1$. Wagon, in [W], proves the result using an angle of arccos(1/3).

Proof. We shall let ϕ and ρ be counterclockwise rotations about the z-axis and x-axis, respectively, each through the angle $\arccos 3/5$. Then

$$\phi^{\pm 1} = \begin{pmatrix} 3/5 & \mp 4/5 & 0 \\ \pm 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \rho^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & \mp 4/5 \\ 0 & \pm 4/5 & 3/5 \end{pmatrix} .$$

We intend to show that no nontrivial reduced word in $\phi^{\pm 1}, \rho^{\pm 1}$ is the identity. For if there is such a word, we may conjugate by ϕ (if necessary) to obtain a word w ending in ϕ which equals the identity.

We claim that $w(1,0,0)$ is of the form $(a,b,c)/5^k$ where a,b,c are integers and b is not divisible by 5. This shows that $w(1,0,0) \neq (1,0,0)$, which is a contradiction.

The claim will be proved by induction. We start off with w of length one. Then $w = \phi^{\pm 1}$, and $w(1,0,0) = (3,4,0)/5$, and everything checks.

Now, for longer words, if $w = \phi^{\pm 1} w'$ or $w = \rho^{\pm 1} w'$, where w' is of the form $(a',b',c')/5^{k-1}$, then $w(1,0,0) = (a,b,c)/5^k$, where

$$a = 3a' \mp 4b', \;\; b = 3b' \pm 4a', \;\; c = 5c' \;\; \text{for } w = \phi^{\pm 1} w',$$

$$\text{or } a = 5a', \;\; b = 3b' \mp 4c', \;\; c = 3a' \pm 4b' \;\; \text{for } w = \rho^{\pm 1} w'.$$

This shows that a,b,c will always be integers.

To show b is never divisible by 5, we consider what happens when w is of the following four forms (where ν is any word):

$$\text{if } w = \phi^{\pm 1}\rho^{\pm 1}\nu, \text{ then } b = 3b' \pm 4a' \text{ where 5 divides } a';$$

$$\text{if } w = \rho^{\pm 1}\phi^{\pm 1}\nu, \text{ then } b = 3b' \mp 4c' \text{ where 5 divides } c';$$

$$\text{if } w = \phi^{\pm 1}\phi^{\pm 1}\nu \text{ or } w = \rho^{\pm 1}\rho^{\pm 1}\nu, \text{ then } b = 6b' - 25b'',$$

where a'', b'', c'' are the integers arising from $\nu(1,0,0)$.

The validity of the first two cases is obvious from the preceding equations; in the last two cases, the arguments are similar and go like this one for the third case: $b = 3b' \pm 4a' = 3b' \pm (12a'' \mp 16b'') = 3b' + 9b'' \pm 12a'' - 16b'' - 9b'' = 3b' + 3(3b'' \pm 4a'') - 25b'' = 6b' - 25b''$.

Q.E.D.

Now every $g \in F$, our free subgroup, fixes two points on the sphere. Let $D = \{$ all points fixed by some $g \in G\}$. Then D is countable, and to $S^2 \setminus D$ we can apply Theorem 6 to obtain:

Theorem 9 (Hausdorff Paradox) *There is a countable set D such that $S^2 \setminus D$ is SO_3-paradoxical.*

The Banach-Tarski Paradox

We shall soon show, however, this countable set D doesn't matter much.

Theorem 10 *S^2 and $S^2 \setminus D$ are SO_3-equidecomposable.*

The proof is very similar to the proof of Theorem 3. Choose an axis of rotation which doesn't fix any points of D. Then consider the set of all rotations around this axis for which some integer multiple of it sends a point of D to another point of D. This is a countable set. But there are uncountably many possible angles; pick one, say θ.

Let ρ_θ be the rotation around the axis by θ. Then no multiple of ρ_θ will send a point of D to another point of D, and it follows that $\rho_\theta^m(D)$ and $\rho_\theta^n(D)$ are disjoint for $m \neq n \geq 0$.

So let $A = \cup\{\rho_\theta^n(D) : n \in \mathbf{N}^+\}$ and $B = S^2 \setminus A$.

We obtain $S^2 \setminus D = B \cup A \sim B \cup \rho_\theta^{-1}(A) = S^2$.

Q.E.D.

This shows that S^2 is SO_3-paradoxical. Moreover, it follows that

Theorem 11 (Banach-Tarski Paradox) *B^3, the solid ball in $\mathbf{R}^3$, is G_3-paradoxical.*

Proof. Since S^2 is paradoxical, we can obtain a paradox for any "thickened" shell by producing a paradox for one sphere in this shell, and then putting points along a radius in the same piece in a decomposition. In particular, we see that $B^3 \setminus \{0\}$ is paradoxical.

It remains to be shown that $B^3 \setminus \{0\} \sim B^3$, for then

$$\begin{aligned} B^3 &= B^3 \setminus \{0\} \ \cup\ \{0\} \\ &\sim B^3 \setminus \{0\} \ \cup\ B^3 \setminus \{0\} \ \cup\ \{0\} \\ &= B^3 \setminus \{0\} \ \cup\ B^3 \\ &\sim B^3 \ \cup\ B^3, \end{aligned}$$

which would prove the theorem.

So choose a broken circle $C' \subseteq B^3 \setminus \{0\}$ with the origin as its broken point. By Theorem 3, C' is equidecomposable with a completed circle C. Thus,

$$\begin{aligned} B^3 \setminus \{0\} &= B^3 \setminus (\{0\} \cup C') \ \cup\ C' \\ &\sim B^3 \setminus (\{0\} \cup C') \ \cup\ (C' \cup \{0\}) \\ &= B^3. \end{aligned}$$

Q.E.D.

Almost an identical proof gives

Corollary 11.1 $\mathbf{R}^3$ *is paradoxical.*

The equidecomposability relation is a useful tool, as we have seen in our proofs, and so it deserves further investigation. It is an equivalence relation, so we can define another relation between $\sim$-equivalence classes of sets: $A \preceq B$ if and only if A is equidecomposable with a subset of B.

This relation $\preceq$ is both reflexive and transitive, by definition. Furthermore, it is also anti-symmetric: if $A \preceq B$ and $B \preceq A$, then $A \sim_G B$. This fact is proved by a generalization of the well-known Schröder-Bernstein Theorem of set theory— Banach was able to generalize it for equivalence relations satisfying these two properties (which $\sim_G$ satisfies):

(i) if $A \preceq B$, then there exists a bijection $g : A \to B$ such that if $C \subseteq A$, then $C \sim g(C)$,

(ii) if $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$, and if $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1 \cup A_2 \sim B_1 \cup B_2$.

Theorem 12 (Banach-Schröder-Bernstein Theorem) *Let G act on X and $A, B \subseteq X$. Then $A \preceq B$ and $B \preceq A$ implies that $A \sim_G B$.*

Proof. This proof is practically identical to the proof of the classical theorem. Notice how the only information we use about $\sim$ are the properties (i) and (ii) above.

Let $f : A \to B'(\subseteq B)$ and $g : B \to A'(\subseteq A)$ be bijections which exist by property (i). Now let $C_0 = A \setminus A_1$, and define inductively $C_{n+1} = gf(C_n)$.

Let $C = \cup C_n$. Then $(A \setminus C) = g(B \setminus f(C))$ since if $x \in f(C)$, then $g(x)$ must be in C. Hence, by property (i) $A \setminus C \sim B \setminus f(C)$. But also $C \sim f(C)$. So by property (ii), $A \setminus C \cup C \sim B \setminus f(C) \cup f(C)$, which means $A \sim B$.

Q.E.D.

To show the power of this theorem we now verify that pieces in a paradoxical decomposition can be taken to be a partition of the set:

Theorem 13 *G acts on X, $E \subseteq X$. Then E is G-paradoxical if and only if there exist disjoint A, B such that $A \cup B = E$ and $A \sim B \sim E$.*

Proof. We already know that if E is G-paradoxical, then we can find disjoint A and B such that $A \sim E \sim B$, so all that remains to be shown is that we can take $A \cup B = E$.

But $E \sim A \subseteq E \setminus B \subseteq E$, so we obtain $A \preceq E \setminus B \preceq E$. Now since $E \sim A$, we conclude that $E \setminus B \sim A$. So take $A' = E \setminus B$ and $B' = B$, both of which are equidecomposable to E, disjoint, and whose union is all of E.

Q.E.D.

Theorem 12 also immediately yields a much stronger form of the Banach-Tarski Paradox :

Theorem 14 (Banach-Tarski Paradox , Strong Form) *If A and B are bounded subsets of $\mathbf{R}^3$ with nonempty interior, then $A \sim B$.*

Proof. We want to show $A \preceq B$, for then $B \preceq A$ is similarly proved, and we obtain by Theorem 12 that $A \sim B$.

So choose balls K and L such that K contains A and L is contained by B. Then for some n large enough, $K \subseteq n$ copies of L. Then we have that $A \subseteq K \subseteq (n \text{ copies of } L) \preceq L \subseteq B$, where the $\preceq$ arises from repeated Banach-Tarski Paradox duplications of L. Hence $A \preceq B$.

Similarly, we obtain $B \preceq A$, and so $A \sim B$.

Q.E.D.

Congruence by Dissection

A problem of classical geometry asks when a polygon can be cut up (in the classical sense, ignoring boundaries) into polygonal pieces and reassembled to form another polygon. This kind of decomposition equivalence is called **congruence by dissection**.

Theorem 15 (Bolyai-Gerwain Theorem) *Two polygons are congruent by dissection if and only if they have the same area.*

Proof. The forward implication is clear. The reverse implication is shown by proving that any polygon can be dissected into finitely many triangles, a triangle is congruent by dissection to a square, and two squares are congruent by dissection to a larger square. Thus every polygon is congruent by dissection to some square of the same area. If two polygons have the same area, then their associated squares are identical, and the polygons are congruent by dissection with each other.

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Higher dimensional analogues of this theorem do not exist. For instance, Dehn proved that a regular tetrahedron in $\mathbf{R}^3$ is *not* congruent by dissection with a cube of the same volume. The Banach-Tarski Paradox , nevertheless, says that they are equidecomposable.

In the plane, we have the following relationship between congruence by dissection and equidecomposability:

Theorem 16 *If P_1, P_2 are polygons which are congruent by dissection , then $P_1 \sim P_2$.*

Proof. The interiors of the pieces of the polygon are obviously equidecomposable. We want to show that the whole polygon is equidecomposable with the interiors of the pieces, and we do this by letting the interiors "absorb" the boundaries of each of the pieces.

The boundaries consist of a finite number (and length) of intervals. So find a disc contained in the polygon, and use a construction similar to Theorem 3, except take a broken annulus in this disc instead of a broken circle. Cutting up each of the intervals as necessary, we can "absorb" these (open, closed, or half-open) intervals by using an (open, closed, or half-open) annulus, which is equidecomposable with a broken annulus by the technique of Theorem 3, and then inserting these intervals.

Since there are only finitely many such intervals, the process terminates.

Q.E.D.

Note, then, that polygons of the same area are equidecomposable. The converse is also true, but we shall not prove it here. The difference between two and three dimensions is highlighted in the fact that in $\mathbf{R}^3$, *any* two polyhedra are equidecomposable!

Minimizing the Pieces

So far, we have not paid attention to how many pieces we actually use to effect a duplication of a ball. In our previous discussion our duplication of the sphere used no more than eight pieces (four from Theorem 5 multiplied by two from Theorem 10); hence, our duplication of the ball used no more than seventeen (by careful application of Theorem 11 we could bring this down to thirteen). One could question why we should even care about pieces at this point, if it weren't for the truly remarkable fact that we only need five!

This is surprisingly low; in fact, the minimum number needed to duplicate *any* set with any group action is four (because otherwise one of the pieces would by itself be congruent to the entire set, which means it *is* the entire set). We will not eke out this minimal answer for the ball just to torture ourselves, for the method we will use will turn out to be applicable in general to certain types of group actions, called *locally commutative* group actions.

Recall that the **stabilizer** of a point x is $Stab(x) = \{\sigma \in G : \sigma(x) = x\}$.

Definition 3 *Let G act on X. This action will be called* **locally commutative** *if for every point $x \in X$ the stabilizer $Stab(x)$ is commutative, i.e., if the subgroup of G which fixes x is commutative.*

Whereas earlier, fixed points of a free group's action posed a problem, the identification of a locally commutative group action will provide us with a new way to deal with them.

First, we will need the following theorem.

Theorem 17 *Let F be the free group generated by σ and τ. Then we can actually partition F into four sets $A_1, ..., A_4$ such that $\sigma(A_2) = A_2 \cup A_3 \cup A_4$ and $\tau(A_4) = A_1 \cup A_2 \cup A_4$.*

Moreover, for any fixed $w \in F$, the partition can be chosen such that w is in the same piece as the identity of F.

In [W], Wagon seems to use an unneccesary argument to prove the theorem—we shorten it below:

Proof. We shall use the same four-piece partition as described in the proof of Theorem 5, where $B(\phi)$ denotes all words beginning on the left with ϕ. Fix some w in F, and let ρ denote the leftmost letter in the word w.

Now suppose, for instance, that $\rho = \tau^{-1}$. (The other cases are similar.) Then we may choose our partition as follows:

$A_1 = B(\sigma)$

$A_2 = B(\sigma^{-1})$

$A_3 = B(\tau) \setminus \{\tau^n : n \in \mathbf{N}^+\}$

$A_4 = B(\tau^{-1}) \cup \{\tau^n : n \in \mathbf{N}^+\} \cup \{1\}$

These sets have the property that $1, w$ are in the same piece, as well as satisfy the relations above: $\sigma(A_2) = A_2 \cup A_3 \cup A_4$ and $\tau(A_4) = A_1 \cup A_2 \cup A_4$.

Q.E.D.

This yields an fruitful corollary which strengthens Theorem 6 for the particular case where G is a free group of rank 2:

Corollary 17.1 *Let F be a free group of rank 2 acting on X without nontrivial fixed points. Then X is F-paradoxical using four pieces.*

Proof. The crucial observation is that lifting a paradox to a set preserves the number of pieces.

Q.E.D.

We also see, then if G is any group with a free non-Abelian subgroup, then it has a free subgroup of rank 2, and by virtue of Corollary 6.1, is paradoxical using four pieces.

We are now in a position to prove the main result about locally commutative actions:

Theorem 18 *Let F be a free subgroup of rank 2 whose action on X is locally commutative. Then X is paradoxical using four pieces.*

Proof. This fact will be proved by defining the partition orbit by orbit so as to satisfy the familiar relations $\sigma(A_2) = A_2 \cup A_3 \cup A_4$ and $\tau(A_4) = A_1 \cup A_2 \cup A_4$. This will give us the desired paradoxical decomposition using four pieces. Let σ, τ generate F.

Note that all F-orbits on X are either composed of nontrivial fixed points, or have no nontrivial fixed points (if x is fixed by w, then $\rho(x)$ is fixed by $\rho w \rho^{-1}$). In the latter case, we partition the orbit as before (see Theorem 6): take a choice set M from each such orbit, and let $P_i = \{g(M) : g \in A_i\}$, where A_i are obtained from any four piece partition guaranteed by Theorem 17.

For an orbit O consisting entirely of fixed points, we must choose a point x carefully. Let w be a nontrivial word of minimal length which fixes a point in O. Then choose x to be a point which w fixes. The idea is to describe all the points in this orbit as the image of a unique representative $v \in F$ and, as before, to lift the paradox from the group onto this orbit, this time in a more refined manner.

Let ρ be the leftmost letter in the word w. We can see that w can not end in ρ^{-1}, else conjugation of w by ρ would produce a smaller word fixing x.

We claim that we can write any point $y \in O$ as $v(x)$, where v does not end in w or in ρ^{-1}, and that such a v is unique. Why? Take a minimal word v such that $y = v(x)$. Then v clearly cannot end in w or w^{-1}, and if it did end in ρ^{-1}, we could take vw, which doesn't end in w or ρ^{-1}.

To show that v is unique, we use local commutativity to show that only powers of w can fix the point x. Else if some z fixed x, $zw = wz$ by local commutativity, and since F is free, z and w must both be powers of some word t (else we would get a nontrivial relation on the letters of F). This immediately gives that z is a power of w, for otherwise some smaller power of t would fix x, contradicting the choice of w.

Then if u and v are both words of the desired form which send x to y, we obtain that $u^{-1}v$ fixes x. So by the argument above, $u^{-1}v$ is a power of w. We can assume it is a positive power, for if not, we can do the argument which follows for $v^{-1}u$. Then either u^{-1} begins with a ρ (meaning u ends in ρ^{-1}) or u^{-1} cancels the first part of v (meaning v ends in a w) or $u = v$, as desired.

Then we put $y = v(x)$ in P'_i according as $v \in A_i$, where the partition is given by Theorem 17, where we ensure that w and the identity lie in the same piece. Notice here we are using possibly *different* partitions for different orbits O, since our partition depends on the word w, which depends on the orbit.

It is a straightforward, but tedious exercise to demonstrate that this construction of P'_i satisfies $\sigma(P'_2) = P'_2 \cup P'_3 \cup P'_4$ and $\tau(P'_4) = P'_1 \cup P'_2 \cup P'_4$. Really the only thing we must take into account is the possibility of muliplying a word in the desired form by a letter and not getting a word of the desired form, i.e., getting something ending in w or ρ^{-1}. But because w and 1 lie in the same piece of the decomposition, we can interchange them when necessary to obtain a new word representation which *is* of the desired form, and is in the same piece as before.

Therefore, we can take $A_i = P_i \cup P'_i$ as our four-piece decomposition of X.

Q.E.D.

Since the action of SO_3, and in particular of its free subgroup F of rank 2, is locally commutative, we have the following result.

Corollary 18.1 *S^2 is SO_3-paradoxical using four pieces, and this is best pos-*

sible.

The converse of Theorem 18 is true, although we shall not prove it here:

Theorem 19 *If X is G-paradoxical using four pieces, then G has a free subgroup of rank 2, whose action on X is locally commutative.*

The proof proceeds by identifying two elements σ, τ which are independent: if $g_1(A_1) \cup g_2(A_2) = X$ and $g_3(A_3) \cup g_4(A_4) = X$, then let $\sigma = g_1^{-1} g_2$, and define $\tau = g_3^{-1} g_4$. These satisfy the stated conditions.

It follows that

Corollary 19.1 *A group G is paradoxical using four pieces if and only if G has a free subgroup of rank 2.*

Proof. This follows from Theorem 19 and Corollary 17.1, noting that a subgroup acts without nontrivial fixed points on G by left multiplication.

Q.E.D.

Let us now apply Theorem 18 to determine the minimal decomposition of a ball.

Theorem 20 *B^3, the solid ball in $\mathbf{R}^3$, is paradoxical using five pieces, and this is best possible.*

Proof. A five piece decomposition is obtained for the unit ball B by using Theorem 18 to obtain a four piece decomposition of the sphere, and thickening the pieces to obtain a four piece decomposition of the open ball minus the center: let C_1, C_2, C_3, C_4 be the such a partition for $\{x : 0 < |x| < 1\}$. Theorem

18 guarantees the relations: $\sigma(C_2) = C_2 \cup C_3 \cup C_4$ and $\tau(C_4) = C_1 \cup C_2 \cup C_4$ where σ, τ generate a free subgroup F of SO_3.

We shall use the "last" spherical shell S, the boundary of B, to produce the extra point to put at the center. We can choose the partition of F, the free subgroup of rank 2, given by the first part of Theorem 5 and note that it satisfies the following relations: $\sigma(A_2) = A_2 \cup A_3 \cup A_4 \cup \{1\}$ and $\tau(A_4) = A_1 \cup A_2 \cup A_4 \cup \{1\}$. (Take $A_1 = B(\sigma), A_2 = B(\sigma^{-1}), A_3 = B(\tau), A_4 = B(\tau^{-1})$.)

We apply this partition to one F-orbit O of S consisting of nonfixed points; for all other orbits in S, use the same partitions as in Theorem 18. Pick any point x in O, and partition O in the usual fashion with this partition: $S_{O,i} = \{g(x) : g \in A_i\}$. We thus obtain a partition of all of S into $S_1, S_2, S_3, S_4, \{x\}$, satisfying $\sigma(S_2) = S_2 \cup S_3 \cup S_4 \cup \{x\}$, and $\tau(S_4) = S_1 \cup S_2 \cup S_4 \cup \{x\}$.

Let ρ be the isometry taking x to 0. Then we can piece everything together; let $B_1 = C_1 \cup S_1 \cup \{0\}$, and let the other $B_i = C_i \cup S_i$. Observe now that $B_1 \cup \sigma(B_2) = B$ and $B_3 \cup \tau(B_4) \cup \rho(x) = B$.

To see that this is best possible, we assume to the contrary. Suppose we could find a four piece paradoxical decomposition of the ball B, witnessed by A_1, A_2, A_3, A_4 such that $\cup A_i = B$, and $\sigma_1(A_1) \cup \sigma_2(A_2) = B$ and $\sigma_3(A_3) \cup \sigma_4(A_4) = B$. Without loss of generality let A_1 be the piece containing $\{0\}$.

Then not both of A_3, A_4 can fix the origin, else the second copy of B would be missing a center point, since neither piece contains the origin. So suppose that $\sigma_3(0) \neq 0$. Let S denote the spherical boundary of B. Then $\sigma_3(A_3) \cap S \subseteq \sigma_3(B) \cap S \subseteq$ an open hemisphere of S, whence $\sigma_4(A_4)$ contains some closed hemisphere H of S.

This implies that A_4 contains a closed hemisphere $\sigma_4^{-1}(H)$ of S, from which we conclude that $(A_1 \cap A_2) \cap S$ is contained in $S \setminus H$, an open hemisphere.

But if either of σ_1, σ_2 did not fix the origin, a similar argument shows that the other of the two pieces A_1, A_2 would contain a closed hemisphere of S, contradicting the above remark.

So *both* σ_1, σ_2 fix the origin, meaning they map S to S. Immediately we have a contradiction, for then $\sigma_1(A_1) \cup \sigma_2(A_2)$ should cover the entire ball B, including S, but each of $\sigma_1(A_1) \cap S, \sigma_2(A_2) \cap S$ is only contained in an open hemisphere by the above remark, and hence cannot cover all of S.

Q.E.D.

m-divisibility

We turn now to a related topic in the study of set decompositions, namely, given a whole number m, is it possible to partition X into m congruent pieces? If it is, we say that X is **m-divisible**. We broach this subject only because we can use an argument similar in flavor to the proof of the previous theorem, to show the following:

Theorem 21 *B^n is not m-divisible for $2 \leq m \leq n$.*

Proof. Suppose B^n is m-divisible. Then let $A_1, ..., A_m$ partition the ball into congruent pieces via $\sigma_i(A_1) = A_i$. We can assume without loss of generality that A_1 is the set containing the center point $\{0\}$. This implies that the σ_i cannot fix the center point, or else the pieces would not be disjoint. Thus, by the argument above, we see that $A_i \cap S = \sigma_i(A_1) \cap S \subseteq \sigma_i(B^n) \cap S \subseteq$ some open hemisphere.

Then this is true for each A_i, including A_1, because $A_i \cap \sigma_i(S)$ lies in an open hemisphere of $\sigma_i(S)$. Therefore S must be coverable by m open hemispheres, which is impossible if $m \le n$. This is easily seen by taking n "poles" of these hemispheres which determine a plane in $\mathbf{R}^n$. This plane cuts S into two pieces, the larger one of which contains a point more than a right angle away from any of the poles. Hence, this point is not covered by any of the open hemispheres.

Q.E.D.

It is unknown whether B^n is m-divisible for $m \ge n$. We do know a few other isolated results on m-divisibility: that S^2 is m-divisible for any $m \ge 3$, and that an open or closed interval is not m-divisible for m finite ≥ 1.

Amenable Groups

Earlier we discussed the implications that paradoxes have for the existence of measures. Now we shall explore a little more this relationship. We shall present a quick (and not at all comprehensive) survey on the work done in this area in relation to paradoxes.

Recall that an amenable group G is a group bearing a left-invariant, finitely additive measure of total measure 1, defined on all subsets of G. Then clearly an amenable group is non-paradoxical, for otherwise the measure will produce the contradiction $1 = 2$.

A somewhat surprising fact is that the converse is also true, which follows from the following result of Tarski.

Theorem 22 (Tarski's Theorem) *Let G act on X, and let $E \subseteq X$. Then there exists a finitely additive, G-invariant measure on X defined for all subsets of X and normalizing E if and only if E is not G-paradoxical.*

In particular, we see that a group G is amenable if and only if it is not paradoxical. For a proof of Tarski's theorem, we refer the reader to [W, pp. 125-128].

The notion of the amenability of a group seems to be quite an intrinsic one, for there are many equivalent definitions of amenability in other contexts. For instance, amenability is equivalent to an assertion about the *cogrowth* function of a group which is finitely presented— loosely speaking, this function counts the number of words of length n which get killed, and the group is amenable if and only if this function in some sense asymptotically approaches the number of total words in the group. This characterization is due to J. Cohen. Another definition of amenability, due to H. Kesten, relates this property to the recurrence of random walks in a group with respect to symmetric probability distributions on the group (see [W, p.157-161] for a reference, and a long list of other equivalent definitions.)

One may ask whether all groups with no free subgroup of rank 2 are amenable. This question was only recently answered in the negative, that is, there is a paradoxical group with no elements of infinite order. However, under certain restrictions, we do obtain that amenability is equivalent to not having a free subgroup— for instance, in the group of isometries of Euclidean space.

Therefore we can show that no paradoxes (using isometries) exist for the balls in $\mathbf{R}^1$ and $\mathbf{R}^2$, because these isometry groups have no free subgroups (since they are solvable, and solvability implies the existence of a finite relation for any elements in the group derived from the commutator chain characterization of solvability).

Thus we have the strange fact that B^n is paradoxical when $n \geq 3$, and *not* paradoxical for $n = 1$ or $n = 2$! Recall, however, that the Sierpiński-

Mazurkiewicz Paradox shows that paradoxes exist in the plane for sets *without* interior; this may be contrasted with the fact than the line has *no* paradoxical subset.

Conclusion

Paradoxes are useful because they challenge the intuition. We have seen that paradoxes were constructed to show the nonexistence of certain measures—but have gained notoriety in their own right as a curiosity. We have only scratched the surface of the literature on paradoxes. For example, we have only considered construction of paradoxes on Euclidean spaces. What about other spaces, such as the hyperbolic plane? Actually, it turns out a Hausdorff Paradox for the hyperbolic plane actually uses Borel pieces! (For the Euclidean plane this cannot happen because of Lebesgue measure.) One open problem asks whether no compact metric space can be paradoxical using Borel pieces. Another (Marczewski's Problem) asks whether the S^2 admits a paradoxical decomposition where the pieces have the property of Baire, i.e., differs from a Borel set by a countable union of nowhere dense sets. Or, can a disc in the plane be split into three congruent pieces?

Clearly such questions will entertain mathematicians for many years to come.

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[W] Wagon, S., *The Banach-Tarski Paradox*, Cambridge Univ. Press., 1986.

Wagon's book will tell you anything and everything you want to know about the Banach-Tarski Paradox, plus much, much more... it is truly an encyclopedic account. The bibliography is extensive.

[W2] Wagon, S., "Partitioning intervals, spheres and balls into congruent pieces", *Can. Math. Bull.*, Vol. 26(1983), pp. 337-340.

This short work discusses m-divisibility.